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Viscous and Joule heating effects on MHD-free convection flow with variable plate temperature

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1. INTRODUCTION

THE NATURAL convection boundary layer flow of an electrically conducting fluid up a hot vertical wall in the presence of a strong magnetic field has been studied by several authors [1–4] because of its application in nuclear engineering in connection with the cooling of reactors. Later, Cramer and Pai [5], presented a similarity solution for the above problem with varying surface temperature. On the other hand Wilks [6] investigated the problem with uniform heat flux illustrating the problem by formulating in terms of regular and inverse series expansions of a characterizing coordinate that provided a link between the similarity states appropriate to the leading edge and downstream. Recently, Hossain and Ahmed [7] have studied a combined effect of forced and free convection with uniform heat flux in the presence of a strong magnetic field. In all the above studies, the effects of both the viscous and Joule heating were neglected because they are of the same order as well as negligibly small (Sparrow and Cess [2]). But Gebhart [8] has shown that the viscous dissipation effect plays an important role in natural convection in various devices which are subjected to large deceleration or which operate at high rotative speeds and also in strong gravitational field processes on large scales (on large planets) and in geological processes. With this understanding Takhar and Soundalgekar [9] have studied the effects of viscous and Joule heating on the problem posed by Sparrow and Cess [2], using the series expansion method of Gebhart [8]. In the present paper, we propose to study the effect of viscous and Joule heating on the flow of an electrically conducting and viscous incompressible fluid past a semi-infinite plate of which temperature varies linearly with the distance from the leading edge and in the presence of uniform transverse magnetic field. The equations governing the flow are solved numerically applying the finite difference method along with Newton's linearization approximation. Numerical solutions are obtained for small Prandtl numbers, appropriate for coolant liquid metal, in the presence of a large magnetic field.

2. BASIC EQUATIONS

The basic equations (1)–(3) describe steady two-dimensional, laminar free convection boundary layer flow of viscous incompressible and conducting fluid through a uniformly distributed transverse magnetic field of strength B_0

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma_0 B_0^2}{\rho}u \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma_0 B_0^2}{\rho c_p}u^2. \tag{3}$$

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Here u and v are the velocity components associated with the direction of increasing coordinates x and y, measured along and normal to the vertical plate, respectively, T is the temperature of the fluid in the boundary layer, g is the acceleration due to gravity, β is the coefficient of thermal expansion, k is the thermal conductivity, ρ is the density of the fluid, σ_0 is the electrical conductivity, ν is the kinematic coefficient of viscosity, c_p is the specific heat at constant pressure and T_∞ is the temperature of the ambient fluid.

The boundary conditions are

$$\left.\begin{aligned} u=v=0,\quad T=T_w(x)\quad &\text{at}\quad y=0\\ u\to 0,\quad T\to T_\infty\quad &\text{as}\quad y\to\infty\end{aligned}\right\}. \tag{4}$$

In formulating equations (1)–(3), it has been assumed that (i) the ratio of thermal diffusivity to magnetic diffusivity is small compared with unity, (ii) fluid property variations are limited to density variation which is taken into account only in so far as it affects the buoyancy terms only and (iii) the short-circuit assumption applies.

To reduce equations (1)–(3) to ordinary differential equations, the stream function ψ defined by $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$ is introduced throughout equations (1)–(3) to get:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=g\beta(T-T_\infty)+\nu\frac{\partial^3\psi}{\partial y^3}-\frac{\sigma_0 B_0^2}{\rho}\frac{\partial\psi}{\partial y} \tag{5}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c_p}\left(\frac{\partial^2\psi}{\partial y^2}\right)^2+\frac{\sigma_0 B_0^2}{\rho c_p}\left(\frac{\partial\psi}{\partial y}\right)^2. \tag{6}$$

We now introduce the following set of transformations. These are defined by

$$\begin{aligned}&\psi(\eta,\xi)=(g\beta N)^{1/4}\nu^{1/2}xf(\eta,\xi),\\ &\eta=(g\beta N)^{1/4}\nu^{-1/2}y\\ &\theta(\eta,\xi)=(T-T_\infty)/(T_w-T_\infty),\quad \xi=g\beta x/c_p\\ &T_w-T_\infty=Nx,\quad M^2=\sigma_0B_0^2/\rho(g\beta N)^{1/2}\end{aligned} \tag{7}$$

substituted into equations (5), (6) and (4), to find

$$f'''+ff''-(M^2+f')f'+\theta=\xi\left\{f'\frac{\partial f'}{\partial\xi}-f''\frac{\partial f}{\partial\xi}\right\} \tag{8}$$

$$\sigma^{-1}\theta''+f\theta'-f'\theta=\xi\left\{f'\frac{\partial\theta}{\partial\xi}-\theta'\frac{\partial f}{\partial\xi}-M^2(f')^2-(f'')^2\right\} \tag{9}$$

$$\left.\begin{aligned}f(0,\xi)=f'(0,\xi)=0,\quad \theta(0,\xi)=1\\ f'(0,\infty)=\theta(0,\infty)=0\end{aligned}\right\}. \tag{10}$$

The physical quantities of interest are the local friction factor τ_w and the local heat transfer q. These are defined by

$$\tau_w=-\rho\nu^{-1/2}(g\beta N)^{3/4}xf''(0,\xi) \tag{11}$$

and

$$q = -kv^{-1/2}(g\beta N)^{1/4}(T_w - T_\infty)\theta'(0,\xi). \quad (12)$$

In equations (7)–(12), the primes denote differentiations of the functions with respect to η.

3. SOLUTIONS AND DISCUSSIONS

Equations (8) and (9) along with the boundary condition (10) have been discretized with a simple implicit finite-difference scheme, similar to that used by Keller and Cebeci [11]. Before we describe equations (8) and (9), we write them in terms of the first-order system of pdes, as given below

$$f' = U \quad (13a)$$

$$U' = V \quad (13b)$$

$$V' + fV - (M^2 + U)U + \theta = \xi\left(U\frac{\partial U}{\partial \xi} - V\frac{\partial f}{\partial \xi}\right) \quad (13c)$$

$$\theta' = W \quad (14a)$$

$$\sigma^{-1}W' + fW - U\theta = \xi\left(U\frac{\partial \theta}{\partial \xi} - W\frac{\partial f}{\partial} - M^2U^2 = V^2\right) \quad (14b)$$

and the boundary conditions turn into

$$\left.\begin{aligned} f(0,\xi) = U(0,\xi) = 0, \quad \theta(0,\xi) = 0 \\ U(\eta_\infty,\xi) = \theta(\eta_\infty,\xi) = 0 \end{aligned}\right\}. \quad (15)$$

At the net rectangle, we denote the net points by

$$\left.\begin{aligned} &\xi_0 = 0, \quad \xi^n = \xi^{n-1} + k_n \quad && n = 1,2,\ldots,N \\ &\eta_0 = 0, \quad \eta_j = n_{j-1} + h_j \quad && j = 1,2,\ldots,J, \quad \eta_J = \eta_\infty \end{aligned}\right\}. \quad (16)$$

Here n and j are just the sequence numbers.

Now, we approximate the quantities (f, U, V, θ, W) at points (ξ^n, η_j) η the net by (f_j^n, $U_j^n \cdot V_j^n$, θ_j^n, W_j^n). We also employ g_j^n for points and quantities midway between net points and for any net function:

$$\left.\begin{aligned} \xi^{n-1/2} \equiv \tfrac{1}{2}(\xi^n + \xi^{n-1}), \quad \eta_{j-1/2} \equiv \tfrac{1}{2}(\eta_j + \eta_{j-1}) \\ g_j^{n-1/2} \equiv \tfrac{1}{2}(g_j^n + g_j^{n-1}), \quad g_{j-1/2}^n \equiv \tfrac{1}{2}(g_j^n + g_{j-1}^n) \end{aligned}\right\}. \quad (17)$$

We now show, the finite-difference approximation of equations (13) and (14) for the mid-point (ξ^n, η_{j-n}), below

$$h_j^{-1}(f_j^n - f_{j-1}^n) = U_{j-1/2}^n \quad (18a)$$

$$h_j^{-1}(U_j^n - U_{j-1}^n) = V_{j-1/2}^n \quad (18b)$$

$$\begin{aligned} h_j^{-1}(\theta_j^n - \theta_{j-1}^n) &= W_{j-1/2}^n h_j^{-1}(V_j^n - V_{j-1}^n) \\ &+ (1+\alpha_n)\{(fV)_{j-1/2}^n - (U^2)_{j-1/2}^n\} - M^2U_{j-1/2}^n \\ &+ \theta_{j-1/2}^n + \alpha_n\{V_{j-1/2}^{n-1}f_{j-1/2}^n - f_{j-1/2}^{n-1}V_{j-1/2}^n\} = R_{j-1/2}^{n-1} \end{aligned} \quad (18c)$$

$$\begin{aligned} \sigma^{-1}h_j^{-1}(W_j^n - W_{j-1}^n) &+ (1+\alpha_n)\{(fW)_{j-1/2}^n - (U\theta)_{j-1/2}^n\} \\ &+ \alpha_n\{\theta_{j-1/2}^{n-1}U_{j-1/2}^n - U_{j-1/2}^{n-1}\theta_{j-1/2}^n + W_{j-1/2}^{n-1}f_{j-1/2}^n \\ &- f_{j-1/2}^{n-1}W_{j-1/2}^n - M^2(U^2)_{j-1/2}^n - (V^2)_{j-1/2}^n\} = T_{j-1/2}^{n-1} \end{aligned} \quad (18d)$$

where $\alpha_n = \xi^{n-1/2}/k_n$

$$\begin{aligned} R_{j-1/2}^{n-1} &= -L_{j-1/2}^{n-1} + \alpha_n\{(fV)_{j-1/2}^{n-1} - (U^2)_{j-1/2}^{n-1}\} \\ L_{j-1/2}^{n-1} &= h_j^{-1}(V_{j-1/2}^{n-1} - V_{j-1/2}^{n-1}) + (fV)_{j-1/2}^{n-1} - M^2U_{j-1/2}^{n-1} \\ &\quad - (U^2)_{j-1/2}^{n-1} + \theta_{j-1/2}^{n-1} \\ T_{j-1/2}^{n-1} &= -M_{j-1/2}^{n-1} + \alpha_n\{(fW)_{j-1/2}^{n-1} - (U\theta)_{j-1/2}^{n-1} \\ &\quad - M^2(U^2)_{j-1/2}^{n-1} - (V^2)_{j-1/2}^{n-1}\} \\ M_{j-1/2}^{n-1} &= \sigma^{-1}h_j^{-1}(W_j^{n-1} - W_{j-1}^{n-1}) + (fW)_{j-1/2}^{n-1} - (U\theta)_{j-1/2}^{n-1}. \end{aligned} \quad (19)$$

The wall and the edge boundary conditions are

$$f_0^n = 0, \quad U_0^n = 0, \quad \theta_0^n = 1, \quad U_J^n = 0, \quad \theta_J^n = 0. \quad (20)$$

If we assume f_j^{n-1}, U_j^{n-1}, V_j^{n-1}, θ_j^{n-1}, and W_j^{n-1} to be known for $0 \leqslant j \leqslant J$, equations (18)–(20) are a system of $5J+5$ equations for the solution of $5J+5$ unknowns (f_j^n, U_j^n, V_j^n, θ_j^n, W_j^n), $j = 0,1,\ldots,J$. These non-linear systems of algebraic equations are linearized by means of Newton's method which then solved in a very efficient manner by using the Keller-box method, discussed by Cebeci and Bradshaw [12] in a simpler way.

The resulting solutions for the velocity and temperature functions are shown graphically in Figs. 1 and 2 and the numerical values for shear-stress and the heat transfer coefficients are presented in Table 1.

The assumptions used to establish the governing equations are particularly appropriate to liquid metals. Moreover, as liquid metals are currently used as coolants in nuclear engineering (Wilks [6]), we have pursued here solutions into the lower Prandtl number range, e.g. 0.05 for lithium, 0.01 for mercury and 0.005 for sodium. In fact, detailed numerical solutions have been obtained for $\sigma = 1$, 0.72, 0.5, 0.1, 0.05, 0.01, 0.005. Associated numerical data are available from the author. As confirmation of satisfactory reconciliation between the previous works and the present analysis, we present the solutions only for $\sigma = 0.05$, 0.01 and 0.005. In Figs. 1 and 2, the solid curves are due to Cramer and Pai [5], in the absence of viscous and Joule heating, which qualitatively agrees with the results of these authors. From Fig. 1(a) we may conclude that the presence of viscous dissipation reduces the flow field. This further reduces owing to the increase in the dissipative heating when the fluid is being heated. A similar situation is also observed from Fig. 1(b), in the case of the temperature field. Figs. 2(a) and (b) represent, respectively, the velocity and temperature field for different values of the Prandtl number in the absence, as well as in the presence, of viscous and Joule heating. From Fig. 2(a) we may conclude that dissipative heat reduces the velocity field more in the lower Prandtl number fluid than that of the higher Prandtl number. On the other hand, in the case of temperature field, the dissipative heat reduces it faster in the higher Prandtl number than that in lower Prandtl number fluid.

We now discuss the effect of viscous and Joule heating on the shear-stress and the rate of heat transfer at the surface of the wall heat. From Table 1 it may be easily concluded that the presence, as well as an increase in dissipation, reduces both the skin-friction and the rate of heat transfer at the surface. This rate of decrease in the skin-friction and the rate of heat transfer due to the presence of viscous and Joule heating slow down with an increase in the magnetic field.

Table 1. Values $f''(0)$ and $\theta'(0)$ for different values of M and ξ when $\sigma = 0.005$

M	ξ	$f''(0)$	$-\theta'(0)$
	0.000	0.81952	0.06152
1	0.005	0.81360	0.07401
	0.010	0.80873	0.08354
	0.000	0.48258	0.04121
2	0.005	0.48078	0.04961
	0.010	0.47918	0.05649
	0.000	0.32868	0.03218
3	0.005	0.32818	0.03737
	0.010	0.32765	0.04178
	0.000	0.24800	0.02704
4	0.005	0.24792	0.03098
	0.010	0.24760	0.03354
	0.000	0.19886	0.02540
5	0.005	0.19883	0.02801
	0.010	0.19874	0.03033

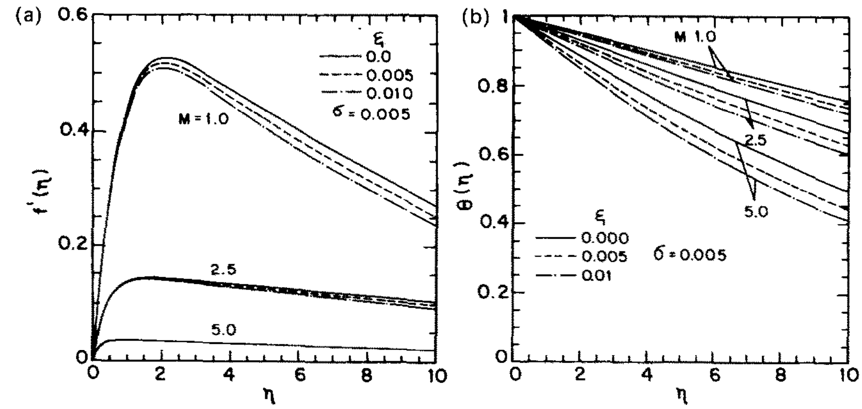

FIG. 1. (a) Velocity profiles against η for different values of ξ and M with $\sigma = 0.005$. (b) Temperature profiles against η for different values of ξ and M with $\sigma = 0.005$.

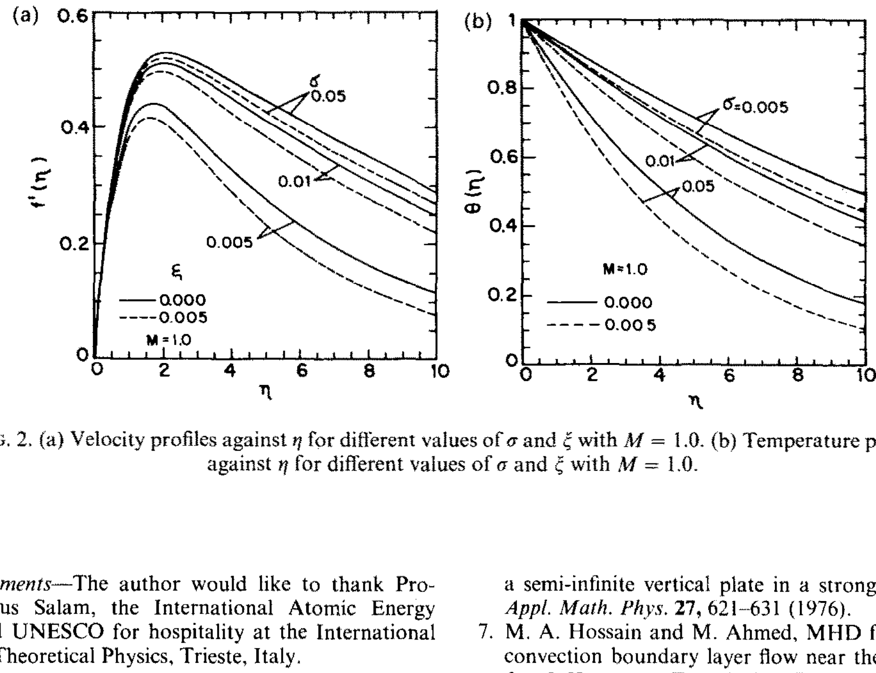

FIG. 2. (a) Velocity profiles against η for different values of σ and ξ with $M = 1.0$. (b) Temperature profiles against η for different values of σ and ξ with $M = 1.0$.

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